# Dynamic potential games: the discrete–time stochastic case

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**Abstract**

This paper concerns a class of nonstationary discrete–time stochastic noncooperative games. Our goals are threefold. First, we characterize Nash equilibria by means of the *Euler equation* approach. Second, we characterize subclasses of *dynamic* potential games. Finally, within one of this subclasses we identify a further subclass for which Nash equilibria are also Pareto (or cooperative) solutions.

**Keywords:** Dynamic games, Euler equation, Potential games, Nash equilibrium, Pareto solution

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## 1 Introduction

This paper concerns a class of nonstationary (or time–varying) discrete–time stochastic noncooperative games. For this class of games, first, we characterize Nash equilibria by means of the so–called *Euler equation* (EE) approach. We use this approach because it is needed to solve some *inverse* control problems that appear in the second part of our work, namely, the characterization of *dynamic* potential games (DPGs). Finally, the latter results are used to identify a subclass of games for which Nash equilibria are also Pareto (or

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cooperative) solutions. In general, of course, Nash equilibria are not Pareto optimal. They do coincide, however, in some special cases. See, for instance, [21] and [26] for differential games.

Roughly speaking, *potential games*—or games with a *potential function*—are noncooperative games whose equilibria can be obtained by means of a single optimization problem. In this optimization problem, the objective function is precisely the potential function.

The concept of potential game was formally introduced in 1973 by Rosenthal [24], but in fact the underlying ideas go back much earlier, for instance, to the 1956 work by Beckmann et al. [5] in the context of congestion games. Nowadays, *static* potential games are well established, with many interesting theoretical results and applications, and there is an extensive literature—see, for instance, [6, 7, 11, 14, 16, 23, 25, 27, 29]. In contrast, *dynamic* potential games have been studied in just a handful of papers and for very particular cases [4, 9, 10, 27].

The remainder of the paper is organized as follows. In Section 2 we introduce the class of dynamic games we are interested in. In Section 3 we review the EE approach to optimal control problems. This approach is used in Section 4 to give conditions for the existence of both Markov–Nash equilibria and open–loop Nash equilibria. The results are illustrated with "the great fish war" of Levhari and Mirman [17]. In Section 5 we move on to potential games. First, to introduce some of the basic ideas, we consider the *static* case. Next, using the results in Section 4, we identify two classes of DPGs which are the natural extension to nonstationary stochastic games of static and/or stationary (or time–invariant) deterministic games [9, 22, 27]. Finally, in Section 6 we characterize a subclass of games for which Nash equilibria are also Pareto solutions. We conclude in Section 7 with some comments on our results as well as some suggestions for further research.

# 2 Nonstationary discrete–time stochastic games

We consider dynamic stochastic games with $n$ players and state space $X \subseteq \mathbb{R}^m$. Let $\{\xi_t\}$ be a sequence of independent random variables, and suppose that each $\xi_t$ takes values in a Borel space $S_t$ $(t = 0, 1, \ldots)$, that is, a Borel subset of a complete and separable metric space. Assume that the state dynamics is given by

$$x_{t+1} = f_t(x_t, u_t^1, \ldots, u_t^n, \xi_t), \quad t = 0, 1, \ldots, \tag{2.1}$$

where $u_t^j$ is chosen by player $j$ in the control set $U^j \subseteq \mathbb{R}^{m_j}$ $(j = 1, \ldots, n)$. In general, the set $U^j$ may depend on time $t$, the current state $x_t$, the action $u_t^i$

of each player $i \neq j$, and the value $s_t$ taken by $\xi_t$, for each $t = 0, 1, \ldots$. We suppose that player $j$ wants to maximize a performance index (also known as reward or payoff function) of the form

$$\mathbb{E} \sum_{t=0}^{\infty} r_t^j(x_t, u_t^1, \ldots, u_t^n) \tag{2.2}$$

subject to (2.1) and the given initial pair $(x_0, s_0)$, which is supposed to be *fixed* throughout the following.

At each time $t = 0, 1, \ldots$, player $j$ chooses an action from the feasible set $U^j$. If each action $u_t^j$ is determined by a Borel–measurable function $\phi^j(t, x_t, s_t)$, where $s_t$ is the realization of $\xi_t$, then we say that $\phi^j$ is a *Markov strategy* for player $j$. In contrast, when $u_t^j = \psi^j(t)$, a function of time $t$ only, then we call $\psi^j$ an *open–loop strategy.* In both cases, the players's decisions are simultaneously and independently chosen. However, in the former case the actions $u_t^j$ are taken after the pair $(x_t, s_t)$ has been observed, whereas the open–loop strategies are decided at the beginning of the game and players commit to follow them.

Recall that the given initial pair $(x_0, s_0)$ is fixed. For player $j$, we denote by $\Phi^j$ and $\Psi^j$ the sets of Markov strategies and open–loop strategies, respectively.

Let $\varphi = (\varphi^1, \ldots, \varphi^n)$ be an $n-$tuple of (Markov or open–loop) strategies. Define

$$V^j(\varphi) := \mathbb{E} \sum_{t=0}^{\infty} r_t^j(x_t, \varphi^1(\cdot), \ldots, \varphi^n(\cdot)), \quad j = 1, \ldots, n,$$

where the state dynamics is given by

$$x_{t+1} = f_t(x_t, \varphi^1(\cdot), \ldots, \varphi^n(\cdot), \xi_t), \quad t = 0, 1, \ldots.$$

**Definition 2.1.** An $n-$tuple $\hat{\phi} = (\hat{\phi}^1, \ldots, \hat{\phi}^n)$ of Borel–measurable Markov strategies is called a *Markov–Nash equilibrium* (MNE) if, for each player $j = 1, \ldots, n$,

$$V^j(\hat{\phi}) \geq V^j(\hat{\phi}^1, \ldots, \hat{\phi}^{j-1}, \phi^j, \hat{\phi}^{j+1}, \ldots, \hat{\phi}^n) \quad \forall \phi^j \in \Phi^j. \tag{2.3}$$

Similarly, an $n-$tuple $\hat{\psi} = (\hat{\psi}^1, \ldots, \hat{\psi}^n)$ of open–loop strategies is an *open–loop Nash equilibrium* (OLNE) if, for each player $j = 1, \ldots, n$,

$$V^j(\hat{\psi}) \geq V^j(\hat{\psi}^1, \ldots, \hat{\psi}^{j-1}, \psi^j, \hat{\psi}^{j+1}, \ldots, \hat{\psi}^n) \quad \forall \psi^j \in \Psi^j. \tag{2.4}$$

The following example illustrates the type of games we are interested in.

**Example 2.2** (The stochastic lake game)**.** Dechert and O'Donnell [10] study the following game about a lake water usage. Communities get benefits from clean lakes, but they also use lakes for drainage through rivers. In this model the state of the system $x_t$ represents the level of phosphorus (typically from fertilizers and animal wastes) in the lake and $u_t^i$ is the level of phosphorus discharged by community $i$ ($i = 1, \ldots, n$) at time $t = 0, 1, \ldots$. The random variable $Z_t$ is assumed to be the amount of rainfall that washes the phosphorus into the lake at period $t$. The state $x_t$ evolves according to

$$x_{t+1} = h(x_t) + (u_t^i + U_{it})Z_t, \quad t = 0, 1, \ldots \tag{2.5}$$

where $U_{it} := \sum_{j \neq i} u_t^j$, and $h$ can be viewed as a natural cleaning function of the lake.

Each community $i$ ($i = 1, \ldots, n$) has a performance index of the form

$$\mathbb{E} \sum_{t=0}^{\infty} \beta^t [v_i(u_t^i) - x_t^2], \tag{2.6}$$

where $v_i(u_t^i)$ is the *utility* derived from loading $u_t^i$, whereas $x_t^2$ is the *disutility* from the effects of phosphorus.

Consider the following optimal control problem (OCP). Given the dynamics (2.5) for the *stochastic lake game* (SLG), maximize

$$\mathbb{E} \sum_{t=0}^{\infty} \beta^t H(u_t^1, \ldots, u_t^n, x_t) \tag{2.7}$$

where $H$ is given by

$$H(u^1, \ldots, u^n, x) := \sum_{i=1}^{n} v_i(u^i) - x^2. \tag{2.8}$$

Dechert and O'Donnell [10] prove that a solution to this OCP is also a Nash equilibrium of the SLG (2.5)–(2.6). Therefore, the SLG is said to be a *dynamic potential game*; see Section 5, below. The function $H$ in (2.7) is called a *potential function* for the SLG.

Dechert and O'Donnell mention that the potential function $H$ in (2.7) can be found by a technique developed in Dechert [8]. Such a technique is an *inverse problem* in deterministic optimal control and it is studied using the so–called *Euler equation* (EE) approach.

In Section 3 we review the EE approach to OCPs, and then in Section 4 we show how to use it to find MNE and OLNE in dynamic games.

# 3 The Euler equation approach

In this section we give conditions to characterize Nash equilibria through the Euler equation approach. For this purpose, we require the following assumptions. For each player $j$, we assume that the control variable $u_t^j$ in (2.1) can be rewritten as a function of the other variables, say

$$u_t^j = h_t(x_t, x_{t+1}, u_t^{-j}, \xi_t), \quad t = 0, 1, \ldots, \tag{3.1}$$

where $u_t^{-j} := (u_t^1, \ldots, u_t^{j-1}, u_t^{j+1}, \ldots, u_t^n)$. If we substitute this expression in (2.2), the performance index for player $j$ becomes of the form

$$\mathbb{E} \sum_{t=0}^{\infty} g_t^j(x_t, x_{t+1}, u_t^{-j}, \xi_t). \tag{3.2}$$

Thus the OCP for player $j$ depends on the control variables of the other players but not on $u^j$. Each player chooses a sequence $\{\hat{x}_t\}$ that maximizes (3.2) given $\{u_t^{-j}\}$. Then the control variable $u_t^j$ is determined by (3.1).

Let us first present the EE approach for OCPs and then, in Section 4, we will extend it to dynamic games.

**Remark 3.1.** If $g : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ is a differentiable function, $\partial g/\partial x$ (or $\partial_x g$) and $\partial g/\partial y$ (or $\partial_y g$) denote the gradients with respect to the first and the second variables, respectively.

Consider the following performance index

$$\mathbb{E} \sum_{t=0}^{\infty} g_t(x_t, x_{t+1}, \xi_t), \tag{3.3}$$

where $\{\xi_t\}$ is a sequence of independent random variables. We suppose that each $\xi_t$ takes values in a Borel space $S_t$ $(t = 0, 1, \ldots)$, that is, a Borel subset of a complete and separable metric space. We also suppose that the initial state $x_0 \in X_0$ and the initial value $\xi_0 = s_0$ are given. Each $x_{t+1}$ has to be chosen in a feasible set $\Gamma_t(x_t, s_t)$ at time $t$ after the value of $\xi_t = s_t$ has been observed. Then we have a family of *feasible sets*

$$\{\Gamma_t(x, s) \subseteq X_{t+1} \mid (x, s) \in X_t \times S_t\}.$$

Let $\varphi = (\mu_0, \mu_1, \ldots)$ be a sequence of measurable functions $\mu_t : X_t \times S_t \to X_{t+1}$ $(t = 0, 1, \ldots)$. For each $(x_0, s_0) \in X_0 \times S_0$, the sequence $\varphi$ determines a Markov process $\{x_t^\varphi \mid t = 1, 2, \ldots\}$ given by

$$\begin{aligned} x_1^\varphi &:= \mu_0(x_0, s_0), \\ x_{t+1}^\varphi &:= \mu_t(x_t^\varphi, \xi_t), \quad t = 1, 2, \ldots. \end{aligned}$$

The sequence $\varphi = (\mu_0, \mu_1, \ldots)$ is said to be a *feasible plan* from $(x_0, s_0)$ if $x_1^\varphi \in \Gamma_0(x_0, s_0)$ and

$$x_{t+1}^\varphi \in \Gamma_t(x_t^\varphi, s) \quad \forall s \in S_t,$$

for $t = 1, 2, \ldots$. The set of all feasible plans from $(x_0, s_0)$ is denoted by $\Phi(x_0, s_0)$.

*The following assumption will be supposed to hold throughout the remainder of the paper.*

**Assumption 3.2.** The stochastic control model

$$(\{X_t\}, \{\xi_t\}, \{g_t\}, \Phi(x_0, s_0)) \tag{3.4}$$

satisfies the following for each $(x_0, s_0) \in X_0 \times S_0$:

(a) the set $\Phi(x_0, s_0)$ is nonempty;

(b) there is a sequence $\{m_t(x_0, s_0)\}$ of nonnegative numbers such that for each $t = 0, 1, \ldots$ and $\varphi \in \Phi(x_0, s_0)$

$$\mathbb{E}[g_t(x_t^\varphi, x_{t+1}^\varphi, \xi_t)] \leq m_t(x_0, s_0),$$

and, moreover, $\sum_{t=0}^\infty m_t(x_0, s_0) < \infty$;

(c) for each $\varphi \in \Phi(x_0, s_0)$, the limit $\lim_{T\to\infty} \mathbb{E} \sum_{t=0}^T g_t(x_t^\varphi, x_{t+1}^\varphi, \xi_t)$ exists (it may be $-\infty$);

(d) there exists $\varphi \in \Phi(x_0, s_0)$ such that $\mathbb{E} \sum_{t=0}^\infty g_t(x_t^\varphi, x_{t+1}^\varphi, \xi_t) > -\infty$.

(e) for each $t = 0, 1, \ldots$ and each possible value $s_t \in S_t$ of $\xi_t$, the function $g_t(\cdot, \cdot, s_t)$ is differentiable in the interior of $X_t \times X_{t+1}$.

For each $(x_0, s_0) \in X_0 \times S_0$, we now define $v : \Phi(x_0, s_0) \to \mathbb{R} \cup \{-\infty\}$ by

$$v(\varphi) := \mathbb{E} \sum_{t=0}^\infty g_t(x_t^\varphi, x_{t+1}^\varphi, \xi_t). \tag{3.5}$$

The stochastic OCP is to find a feasible plan $\hat{\varphi} \in \Phi(x_0, s_0)$ such that

$$\sup\{v(\varphi) \mid \varphi \in \Phi(x_0, s_0)\} = v(\hat{\varphi}). \tag{3.6}$$

Since a feasible plan $\varphi$ is, by definition, a sequence of measurable functions, the set

$$\Lambda := \{(\mu_0, \mu_1, \ldots) \mid \mu_t : X_t \times S_t \to \mathbb{R}^n,\ t = 0, 1, \ldots\}$$

is a vector space. We can use Gâteaux differentials to find necessary conditions on an optimal plan. Thus we need the following assumptions.

**Assumption 3.3.** Let $\hat{\varphi} \in \Phi(x_0, s_0)$ be some internal plan in the direction $\varphi \in \Lambda$; that is, there exists $\varepsilon_0 > 0$ such that

$$\hat{\varphi} + \varepsilon\varphi \in \Phi(s_0, x_0) \quad \forall\, \varepsilon \in (-\varepsilon_0, \varepsilon_0).$$

Denote by $\{\hat{x}_t^{\varphi}\}$ and $\{x_t^{\varphi}\}$ the corresponding state (Markov) processes induced by $\hat{\varphi}$ and $\varphi$, respectively. Define

$$h_0(\varepsilon) := g_0(x_0, \hat{x}_1^{\varphi} + \varepsilon x_1^{\varphi}, s_0), \quad h_t(\varepsilon) := g_t(\hat{x}_t^{\varphi} + \varepsilon x_t^{\varphi}, \hat{x}_{t+1}^{\varphi} + \varepsilon x_{t+1}^{\varphi}, \xi_t),$$

for $t = 1, 2, \ldots$. There exists $\varepsilon_0 > 0$ such

$$\frac{d}{d\varepsilon}\mathbb{E}\sum_{t=0}^{\infty} h_t(\varepsilon) = \mathbb{E}\sum_{t=0}^{\infty}\frac{dh_t}{d\varepsilon}(\varepsilon) \quad \forall\varepsilon \in (-\varepsilon_0, \varepsilon_0).$$

**Assumption 3.4.** There exists an optimal plan $\hat{\varphi} \in \Phi(x_0, s_0)$ for the OCP (3.4)–(3.6). Moreover, for every sequence $\{\xi_t = s_t\}$ of observed values:

(a) $\hat{x}_{t+1}^{\varphi}$ is an interior point of the set $\Gamma_t(\hat{x}_t{}^{\varphi}, s_t)$;

(b) there exists $\varepsilon_t > 0$ such that $\|x - \hat{x}_t^{\varphi}\| < \varepsilon_t$ implies $\hat{x}_{t+1} \in \Gamma_t(x, s_t)$.

The following Theorem 3.5 gives two necessary conditions for an optimal plan, namely, the *stochastic Euler equation* (SEE) (3.7) and the *transversality condition* (TC) (3.8). For a proof of Theorem 3.5 see, for instance, González–Sánchez and Hernández–Lerma [13]. (See also the Remark 3.6.)

Recall the notation in Remark 3.1.

**Theorem 3.5.** *Let $\hat{\varphi} \in \Phi(x_0, s_0)$ be an optimal plan for the OCP* (3.4)–(3.6). *Suppose that Assumption 3.4 holds. Then:*

*(a) $\hat{\varphi}$ satisfies the stochastic Euler equation (SEE)*

$$\mathbb{E}\Big[\frac{\partial g_{t-1}}{\partial y}(\hat{x}_{t-1}^{\varphi}, \hat{x}_t^{\varphi}, \xi_{t-1}) + \frac{\partial g_t}{\partial x}(\hat{x}_t^{\varphi}, \hat{x}_{t+1}^{\varphi}, \xi_t)\Big] = 0, \quad t = 1, 2, \ldots. \tag{3.7}$$

*(b) Suppose that, in addition, $\hat{\varphi}$ and $\varphi \in \Lambda$ satisfy Assumption 3.3. Then $\hat{\varphi}$ and $\varphi$ satisfy the transversality condition (TC)*

$$\lim_{t\to\infty}\mathbb{E}\Big[\frac{\partial g_{t-1}}{\partial y}(\hat{x}_{t-1}^{\varphi}, \hat{x}_t^{\varphi}, \xi_{t-1}) \cdot x_t^{\varphi}\Big] = 0. \tag{3.8}$$

**Remark 3.6.** In this stochastic model it is assumed that $x_t$ is chosen after the value $\xi_{t-1} = s_{t-1}$ has been observed, for $t = 1, 2, \ldots$. Taking into account this assumption, the SEE (3.7) is also written as

$$\frac{\partial g_{t-1}}{\partial y}(\hat{x}^{\varphi}_{t-1}, \hat{x}^{\varphi}_{t}, s_{t-1}) + \mathbb{E}\Big[\frac{\partial g_t}{\partial x}(\hat{x}^{\varphi}_{t}, \hat{x}^{\varphi}_{t+1}, \xi_t)\Big] = 0, \quad t = 1, 2, \ldots. \tag{3.9}$$

On the other hand, if the optimal plan $\hat{\varphi}$ is an internal plan in the direction $\varphi = \hat{\varphi}$, then the TC (3.8) becomes

$$\lim_{t\to\infty} \mathbb{E}\Big[\frac{\partial g_t}{\partial x}(\hat{x}^{\varphi}_{t}, \hat{x}^{\varphi}_{t+1}, \xi_t) \cdot \hat{x}^{\varphi}_{t}\Big] = 0. \tag{3.10}$$

The TC (3.10) is better known than (3.8). Actually, the TC (3.10) is used (along with (3.9) and Assumption 3.7) as a sufficient condition —Theorem 3.8, below. See, for instance, Acemoglu [3, Section 16.3], González–Sánchez and Hernández–Lerma [13] or Stokey and Lucas [28, Section 9.5].

**Assumption 3.7.** The OCP (3.4)–(3.6) satisfies the following:

(a) $g_t(\cdot, \cdot, s)$ is concave for each $s \in S_t$;

(b) the set $\Phi(x_0, s_0)$ is convex;

(c) $X_t$ is a subset of $\mathbb{R}^m_+ = \{(x_1, \ldots, x_m) \mid x_k \geq 0,\ k = 1, \ldots, m\}$;

(d) for each $k = 1, 2, \ldots, m$, $\partial g_t/\partial x_k \geq 0$.

**Theorem 3.8.** *Suppose that a feasible plan $\hat{\varphi} \in \Phi(x_0, s_0)$ satisfies the SEE* (3.7) *and the TC* (3.10)*. If Assumption 3.7 holds, then $\hat{\varphi}$ is an optimal plan for the OCP* (3.4)*–*(3.6)*.*

In the following section we show how to extend the EE approach to dynamic games.

# 4 Characterization of Nash equilibria

We now go back to the stochastic games introduced in Section 2. We wish to use the EE approach to characterize Markov–Nash equilibria (MNE) as well as open–loop Nash equilibria (OLNE).

## 4.1 Markov–Nash equilibria

We first assume that players take their actions according to Markov strategies, say $u_t^j = \phi^j(t, x_t, s_t)$. By (2.1), a Markov multi–strategy $\phi = (\phi^1, \ldots, \phi^n)$ induces a Markov process $\{x_t^\phi\}$. Thus, each function $g_t^j$ in (3.2) depends on $(x_t^\phi, x_{t+1}^\phi, \xi_t)$ only. More precisely, with a Markov multi–strategy $\phi = (\phi^1, \ldots, \phi^n)$, each player $j$ has a performance index of the form

$$\mathbb{E} \sum_{t=0}^{\infty} G_t^j(x_t^\phi, x_{t+1}^\phi, \xi_t), \tag{4.1}$$

where $G_t^j(x, y, s) := g_t^j(x, y, \phi^{-j}(t, x, s), s)$ and

$$\phi^{-j} := (\phi^1, \ldots, \phi^{j-1}, \phi^{j+1}, \ldots, \phi^n).$$

Therefore, for each $j = 1, \ldots, n$, there is a control model as in (3.4). It is assumed that each of these control models satisfies Assumption 3.2.

We can specify a game in a reduced form as

$$(\{X_t\}, \{\xi_t\}, \{G_t^j | j \in J\}, \{\Phi^j | j \in J\}), \tag{4.2}$$

where $J = \{1, \ldots, n\}$ is the set of players.

**Assumption 4.1.** The game model (4.2) satisfies the following for each $j = 1, \ldots, n$ and $t = 0, 1, \ldots$:

(a) $g_t^j(\cdot, \cdot, \cdot, s)$ and $\phi^j(t, \cdot, s)$ are differentiable for each $s \in S_t$, and so $G_t^j(x, y, s)$ is differentiable in $(x, y)$;

(b) $G_t^j(x, y, s)$ is concave in $(x, y)$ for each $s \in S_t$;

(c) the set $\Phi^j$ is convex;

(d) $X_t$ is a subset of $\mathbb{R}_+^m = \{(x_1, \ldots, x_m) \mid x_k \geq 0, \ k = 1, \ldots, m\}$;

(e) for each $k = 1, 2, \ldots, m$, $\partial G_t^j / \partial x_k \geq 0$.

The following theorem follows from Theorem 3.8.

**Theorem 4.2.** *Let $\phi = (\phi^1, \ldots, \phi^n)$ be a Markov multi–strategy and let $\{x_t^\phi\}$ be the induced Markov process. Suppose that $\{x_t^\phi\}$ satisfies, for each $j = 1, \ldots, n$, the SEE*

$$\frac{\partial G_{t-1}^j}{\partial y}(x_{t-1}^\phi, x_t^\phi, s_{t-1}) + \mathbb{E} \frac{\partial G_t^j}{\partial x}(x_t^\phi, x_{t+1}^\phi, \xi_t) = 0, \quad t = 1, 2, \ldots. \tag{4.3}$$

*and the TC*

$$\lim_{t \to \infty} \mathbb{E} \Big[ \frac{\partial G_t^j}{\partial x}(x_t^\phi, x_{t+1}^\phi, \xi_t) \cdot x_t^\phi \Big] = 0. \tag{4.4}$$

*If Assumption 4.1 holds, then $(\phi^1, \ldots, \phi^n)$ is a MNE for the game (4.2).*

## 4.2 Deterministic open–loop equilibria

**Remark 4.3.** One of the main assumptions to use the EE approach is that each control variable $u_t^j$ can be put (from (2.1)) in the form (3.1). If the action is chosen by an open–loop strategy, say $u_t^j = \psi^j(t)$, then the right hand side of (3.1) (which is *stochastic*) equals a *deterministic* control. Therefore when following the EE approach, it is **not** possible to consider open–loop strategies in stochastic games.

By virtue of the previous remark, we consider the deterministic version of (2.1), (2.2), and (3.1). Then player $j$ wants to maximize a performance index of the form

$$\sum_{t=0}^{\infty} g_t^j(x_t, x_{t+1}, u_t^{-j}), \tag{4.5}$$

for $j = 1, \ldots, n$. We specify this game model as

$$(\{X_t\}, \{g_t^j | j \in J\}, \{\Psi^j | j \in J\}), \tag{4.6}$$

where $\Psi^j$ denotes the set of open–loop strategies for player $j$. We need the following hypotheses to obtain OLNE.

**Assumption 4.4.** The game model (4.6) satisfies the following for each $j = 1, \ldots, n$ and $t = 0, 1, \ldots$:

(a) $g_t^j(x, y, z)$ is concave and differentiable in $(x, y)$ for each $z$;

(b) the set $\Psi^j$ is convex;

(c) $X_t$ is a subset of $\mathbb{R}_+^m = \{(x_1, \ldots, x_m) \mid x_k \geq 0,\ k = 1, \ldots, m\}$;

(d) for each $k = 1, 2, \ldots, m$, $\partial g_t^j / \partial x_k \geq 0$.

The next theorem also follows from Theorem 3.8.

**Theorem 4.5.** *Let $\psi = (\psi^1, \ldots, \psi^n)$ be an open–loop multi–strategy and let $\{x_t^\psi\}$ be the corresponding state sequence. Suppose that $\{x_t^\psi\}$ satisfies, for each $t = 1, 2, \ldots$ and $j = 1, \ldots, n$, the EE*

$$\frac{\partial g_{t-1}^j}{\partial y}(x_{t-1}^\psi, x_t^\psi, \psi^{-j}(t-1)) + \frac{\partial g_t^j}{\partial x}(x_t^\psi, x_{t+1}^\psi, \psi^{-j}(t)) = 0, \tag{4.7}$$

*and the TC*

$$\lim_{t\to\infty} \frac{\partial g_t^j}{\partial x}(x_t^\psi, x_{t+1}^\psi, \psi^{-j}(t)) \cdot x_t^\psi = 0. \tag{4.8}$$

*If Assumption 4.4 holds, then $(\psi^1, \ldots, \psi^n)$ is an OLNE for the game (4.6).*

## 4.3 Examples: the great fish war

We will now illustrate Theorems 4.2 and 4.5 by means of "the great fish war" of Levhari and Mirman [17]. First we illustrate Theorem 4.5.

**Example 4.6.** The following dynamic game, due to Levhari and Mirman [17], concerns fisheries. Let $x_t$ $(t = 0, 1, 2, \ldots)$ be the stock of fish at time $t$, in a specific fishing zone. Assume there are $n$ countries deriving utility from fish consumption. More precisely, country $i$ wants to maximize

$$\sum_{t=0}^{\infty} \beta_i^t \log(c_t^i), \quad i = 1, \ldots, n,$$

where $\beta_i$ is a discount factor and $c_t^i$ is the consumption corresponding to country $i$. The fish population follows the dynamics

$$x_{t+1} = (x_t - c_t^1 - \ldots - c_t^n)^\alpha, \quad t = 0, 1, 2, \ldots, \tag{4.9}$$

where $x_0$ is given and $0 < \alpha < 1$.

Given an open–loop multi–strategy $\psi = (\psi^1, \ldots, \psi^n)$, the functions $g_t^j$ are

$$g_t^j(x, y, \psi^{-j}(t)) = \beta_j^t \log(x - y^{1/\alpha} - \sum_{i \neq j} \psi^i(t)),$$

for $j = 1, \ldots, n,\ t = 0, 1, \ldots$. Then the EE (4.7) becomes, for each $t = 0, 1, \ldots$,

$$\frac{-x_t^{1/\alpha - 1}/\alpha}{x_{t-1} - x_t^{1/\alpha} - \sum_{i \neq j} \psi^i(t-1)} + \beta_j \frac{1}{x_t - x_{t+1}^{1/\alpha} - \sum_{i \neq j} \psi^i(t)} = 0. \tag{4.10}$$

This system of difference equations has infinitely many solutions as we will see in Example 6.3. Then this is a game with infinitely many OLNE, since the hypotheses of Theorem 4.5 hold.

The following example illustrates Theorem 4.2 which is, of course, also valid in the deterministic case.

**Example 4.7.** Consider again the great fish war game described in Example 4.6. Suppose that player $j$ follows the *stationary* Markov strategy $c_t^j = \phi^j(x_t)$, for $j = 1, \ldots, n$ and $t = 0, 1, \ldots$. Thus, by (4.9) the dynamics is given by

$$x_{t+1} = (x_t - \phi^1(x_t) - \ldots - \phi^n(x_t))^\alpha, \quad t = 0, 1, \ldots. \tag{4.11}$$

Given the strategies $\phi^i$, for $i \neq j$, player $j$ wants to maximize

$$\sum_{t=0}^{\infty} G_t^j(x_t^\phi, x_{t+1}^\phi) = \sum_{t=0}^{\infty} \beta_j^t \log(x_t - x_{t+1}^{1/\alpha} - \sum_{i\neq j} \phi^i(x_t)).$$

For ease of notation we write $x_t$ instead of $x_t^\phi$. Therefore, the SEE (4.3) becomes

$$\frac{-x_t^{1/\alpha-1}/\alpha}{x_{t-1} - x_t^{1/\alpha} - \sum_{i\neq j} \phi^i(x_{t-1})} + \beta_j \frac{1 - \sum_{i\neq j}(\phi^i)'(x_t)}{x_t - x_{t+1}^{1/\alpha} - \sum_{i\neq j} \phi^i(x_t)} = 0 \qquad (4.12)$$

for all $t = 0, 1, \ldots$. Let us try with linear strategies, that is, $\phi^j(x) = a_j x$. Substituting these strategies in (4.11) and (4.12), we get the following (linear) equation for the constants $a_j$ $(j = 1, \ldots, n)$

$$\alpha\beta_j(1 - a_{-j}) = 1 - a, \quad j = 1, \ldots, n, \qquad (4.13)$$

where $a := a_1 + \ldots + a_n$ and $a_{-j} := a - a_j$. The solution to (4.13) is given by

$$a_j = \frac{1 - \alpha\beta_j}{\alpha\beta_j[1 + \Sigma_{i=1}^n (1 - \alpha\beta_i)(\alpha\beta_i)^{-1}]}, \quad j = 1, \ldots, n. \qquad (4.14)$$

It is easy to check that $\phi = (\phi^1, \ldots, \phi^n)$ verifies the hypotheses of Theorem 4.2. Hence the linear multi–strategy $\phi$ is a Markov–Nash equilibrium.

# 5 Potential games

In this section we consider games with dynamics (2.1) and reward function (2.2). Both MNE and OLNE are considered.

**Definition 5.1.** A dynamic game is said to be a *dynamic potential game* (DPG) if there exists an OCP such that a solution to the OCP is also a Nash equilibrium for the game. A dynamic potential game is called a *Markov potential game* (or *open–loop potential game*) if only Markov (or open–loop, respectively) multi–strategies are considered for the game as well as in the corresponding OCP.

The main objective of this section is to find conditions to characterize DPGs. In Theorem 5.9 we characterize a class of DPGs with MNE. In Theorems 5.12 and 6.1 we consider OLNE. First, however, we shall consider *static* games to motivate the kind of results we would expect to obtain.

## 5.1 Static potential games

Consider a static game in normal form, that is, a triplet

$$(\mathcal{N}, \{A_j \mid j \in \mathcal{N}\}, \{u_j \mid j \in \mathcal{N}\}), \tag{5.1}$$

where

(a) $\mathcal{N} = \{1, 2, \ldots, n\}$ is the set of players;

(b) $A_j$ is the action set for player $j \in \mathcal{N}$;

(c) $u_j : A \to \mathbb{R}$ is the payoff function for player $j$, with $A := A_1 \times \ldots \times A_n$.

We assume that each player wants to maximize his/her own payoff function. Therefore we say that $\hat{a} \in A$ is a *Nash equilibrium* for the game (5.1) if for each player, $j = 1, \ldots, n$,

$$u_j(\hat{a}_1, \ldots, \hat{a}_n) \geq u_j(\hat{a}_1, \ldots, \hat{a}_{j-1}, a_j, \hat{a}_{j+1}, \ldots, \hat{a}_n) \quad \forall a_j \in A_j. \tag{5.2}$$

**Remark.** Given $a = (a_1, \ldots, a_n)$ and $j \in \mathcal{N}$, we use $a^{-j}$ to denote

$$(a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n).$$

Analogously, $A_{-j}$ denotes $A_1 \times \ldots \times A_{j-1} \times A_{j+1} \times \ldots \times A_n$. Hence $a^{-j}$ is an element of $A_{-j}$. With an abuse of notation, we write $(a'_j, a^{-j})$ instead of

$$(a_1, \ldots, a_{j-1}, a'_j, a_{j+1}, \ldots, a_n).$$

With this notation, the inequality (5.2) can be written as $u_j(\hat{a}) \geq u_j(a_j, \hat{a}^{-j})$.

From (5.2), we observe that $\hat{a}_j$ maximizes $u_j$ given $\hat{a}^{-j}$ for each $j$. That is, $\hat{a}$ solves $n$ constrained and simultaneous maximization problems. The main question here is: *is there a function $P$, depending on $(a_1, \ldots, a_n)$, such that a maximum of $P$ is also a Nash equilibrium for the game (5.1)?*

The following definitions address this question.

**Definition 5.2** (Monderer and Shapley [22])**.** The game (5.1) is called a *potential game* if there exists a function $P : A \to \mathbb{R}$ such that, for each $j \in \mathcal{N}$ and $a^{-j} \in A_{-j}$,

$$u_j(x, a^{-j}) - u_j(y, a^{-j}) = P(x, a^{-j}) - P(y, a^{-j}) \qquad \forall\, x, y \in A_j.$$

The function $P$ is said to be a *potential* for the game (5.1).

Definition 5.2 does not require any property on the payoff functions nor about the action sets. However, when each $A_j$ is an interval of real numbers, derivatives can be used to maximize $u_j$ $(j \in \mathcal{N})$.

**Definition 5.3** (Slade [27])**.** A differentiable function $P : A \to \mathbb{R}$ is a *fictitious–objective function* for (5.1) if, for every $j \in \mathcal{N}$,

$$\frac{\partial P}{\partial a_j}(a) = \frac{\partial u_j}{\partial a_j}(a) \qquad \forall a \in A.$$

Theorems 5.4 and 5.5, below, answer the question on the existence of a function $P$ whenever $A_j$ is an interval of real numbers for each $j \in \mathcal{N}$.

**Theorem 5.4** (Monderer and Shapley [22])**.** *Suppose that $u_j$ is of class $\mathcal{C}^2$ for each $j \in \mathcal{N}$. Then* (5.1) *is a potential game if and only if*

$$\frac{\partial^2 u_j}{\partial a_j \partial a_i} = \frac{\partial^2 u_i}{\partial a_i \partial a_j} \quad \forall\, i, j \in \mathcal{N}. \tag{5.3}$$

*Moreover, if* (5.3) *holds, and $x_0 \in A$ is arbitrary (but fixed), then a potential function for* (5.1) *is given by*

$$P(x) := \int_0^1 \sum_{j=1}^{n} \frac{\partial u_j}{\partial a_j}(\gamma(t))\gamma_j'(t)\, dt, \tag{5.4}$$

*where $\gamma : [0, 1] \to A$ is a piecewise continuously differentiable path such that $\gamma(1) = x$ and $\gamma(0) = x_0$.*

**Theorem 5.5** (Slade [27])**.** *Suppose that $u_j$ is of class $\mathcal{C}^1$ for each $j \in \mathcal{N}$. Then the following statements are equivalent:*

(a) *A function $P : A \to \mathbb{R}$ is a fictitious–objective function for the game* (5.1).

(b) *For every $j \in \mathcal{N}$, there is a function $f_j : A_{-j} \to \mathbb{R}$ such that*

$$u_j(a) = P(a) + f_j(a^{-j}) \qquad \forall a \in A. \tag{5.5}$$

## 5.2 Markov potential games

**Assumption 5.6.** The game model (4.2), i.e.,

$$(\{X_t\}, \{\xi_t\}, \{G_t^j | j \in J\}, \{\Phi^j | j \in J\}) \tag{5.6}$$

satisfies:

(a) the number of components of the state variables is $n$, the number of players, in particular $X_t \subseteq \mathbb{R}^n$; the state space $X_t$ is nonempty, connected, and open;

(b) $\{\xi_t\}$ is a sequence of independent random variables;

(c) the set $\Phi := \Phi^1 \times \cdots \times \Phi^n$ consists of all Markov multi–strategies where player $j$ decides $x^j_{t+1}$ as a function of $(t, x^j_t, s_t)$;

(d) $G^j_t(x, y, s)$ is of class $\mathcal{C}^2$ in $(x, y)$ for each $s$.

With the reward functions $\{G^j_t | j \in J\}$ in (5.6), define

$$a_{t-1}(x, y, \xi_{t-1}) := \Big[\frac{\partial G^1_{t-1}}{\partial y^1}(x, y, \xi_{t-1}), \ldots, \frac{\partial G^n_{t-1}}{\partial y^n}(x, y, \xi_{t-1})\Big], \quad (5.7)$$

$$b_t(x, y, \xi_t) := \Big[\frac{\partial G^1_t}{\partial x^1}(x, y, \xi_t), \ldots, \frac{\partial G^n_t}{\partial x^n}(x, y, \xi_t)\Big] \quad (5.8)$$

for $t = 1, 2, \ldots$.

**Remark 5.7.** If $f : \mathbb{R}^n \to \mathbb{R}^n$ is measurable with component functions $f_1, f_2, \ldots, f_n$ and $\phi : [0, 1] \to \mathbb{R}^n$ is a $\mathcal{C}^1$ function with components $\phi_1, \phi_2, \ldots, \phi_n$, then

$$\int_{\phi(0)}^{\phi(1)} f(x)dx := \int_0^1 \left[\sum_{i=1}^n f_i(\phi(t))\frac{d\phi_i}{dt}(t)\right] dt.$$

The function $f$ is said to be *exact* when this integral does not depend on the path $\phi$. A necessary and sufficient condition for a $\mathcal{C}^1$ function $f$ to be exact is that

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i} \quad \text{for } i, j = 1, \ldots, n.$$

**Assumption 5.8.** *For each $t = 1, 2, \ldots,$ the functions in (5.7)–(5.8) satisfy the following:*

(a) $\partial_x a_t(x, y, \xi_t) = [\partial_y b_t(x, y, \xi_t)]^*$*, where* $B^*$ *denotes the transpose of a matrix* $B$*;*

(b) $a_{t-1}(x, \cdot, \xi_{t-1})$ *and* $b_t(\cdot, y, \xi_t)$ *are both exact.*

In the following theorem we use the stochastic inverse OCP in González–Sánchez and Hernández–Lerma [12] (which is a stochastic version of Dechert [8]) to characterize DPGs. Dynamic potential games in the deterministic case are also studied in Dechert [9].

**Theorem 5.9.** *Suppose that the game (5.6) satisfies Assumptions 5.6 and 5.8. Then the game (5.6) is a DPG within the multi–strategies as in Assumption 5.6(c).*

*Proof.* Since the functions $\{a_{t-1}\}, \{b_t\}$ defined by (5.7)–(5.8) satisfy Assumption 5.8, then there exists a performance index of the form

$$\mathbb{E}\sum_{t=0}^{\infty} H_t(x_t, x_{t+1}, \xi_t), \tag{5.9}$$

with the functions $H_t$ ($t = 0, 1, \ldots$) given by

$$H_0(x, y, \xi) := \int_{\overline{y}}^{y} a_0(x, w, \xi)\, dw, \tag{5.10}$$

and for $t = 1, 2, \ldots$

$$H_t(x, y, \xi) := \int_{\overline{y}}^{y} a_t(\overline{x}, w, \xi)\, dw + \int_{\overline{x}}^{x} b_t(w, y, \xi)\, dw, \tag{5.11}$$

for some constants $\overline{x}$ and $\overline{y}$. See [12] or [8] for further details.

Observe that

$$\partial_x \int_{\overline{y}}^{y} a_t(x, w, \xi)\, dw = \int_{\overline{y}}^{y} [\partial_x a_t(x, w, \xi)]^* \, dw \tag{5.12}$$

and

$$\partial_y \int_{\overline{x}}^{x} b_t(w, y, \xi)\, dw = \int_{\overline{x}}^{x} [\partial_y b_t(w, y, \xi)]^* \, dw, \tag{5.13}$$

since $a_t$ and $b_t$ are functions of class $\mathcal{C}^1$.

For $t = 1, 2, \ldots$, we claim that the difference

$$\begin{aligned} H_t(x, y, \xi) - H_t(x', y', \xi) \quad &= \int_{\overline{y}}^{y} a_t(\overline{x}, w, \xi)\, dw + \int_{\overline{x}}^{x} b_t(w, y, \xi)\, dw \\ &\quad - \int_{\overline{y}}^{y'} a_t(\overline{x}, w, \xi)\, dw - \int_{\overline{x}}^{x'} b_t(w, y', \xi)\, dw \end{aligned}$$

does not depend on the constants $\overline{x}, \overline{y}$. First, note that the right–hand side equals

$$\int_{y'}^{y} a_t(\overline{x}, w, \xi)\, dw + \int_{\overline{x}}^{x} b_t(w, y, \xi)\, dw - \int_{\overline{x}}^{x'} b_t(w, y', \xi)\, dw, \tag{5.14}$$

which is independent of $\overline{y}$. Moreover, taking the gradient with respect to $\overline{x}$ in (5.14), and using (5.13) and Assumption 5.8(a), we have

$$\int_{y'}^{y} [\partial_y b_t(\overline{x}, w, \xi)]^* \, dw - b_t(\overline{x}, y, \xi) + b_t(\overline{x}, y', \xi) \equiv 0,$$

that is, the difference $H_t(x,y,\xi) - H_t(x',y',\xi)$ does not depend on $\overline{x}$. Therefore, by setting $\overline{x} = x'$ in (5.14), we have

$$\begin{aligned} H_t(x,y,\xi) - H_t(x',y',\xi) &= \int_{y'}^{y} a_t(x',w,\xi)\,dw + \int_{x'}^{x} b_t(w,y,\xi)\,dw \\ &= \sum_{j=1}^{n} [G_t^j(x,y,\xi) - G_t^j(x',y',\xi)]. \qquad (5.15) \end{aligned}$$

Consider an OCP with the same components of (5.6) except that we now consider the performance index (5.9)–(5.11). We will prove that a solution to this OCP is also a Nash equilibrium for the game.

Let $\hat{\phi} = (\hat{\phi}^1, \ldots, \hat{\phi}^n)$ be a solution to the OCP and denote by $\{\hat{x}_t\}$ the corresponding Markov process. Fix a player $j$ and define

$$\widetilde{\phi} = (\hat{\phi}^1, \ldots, \hat{\phi}^{j-1}, \phi^j, \hat{\phi}^{j+1}, \ldots, \hat{\phi}^n),$$

for some strategy $\phi^j \in \Phi^j$. Denote by $\{\widetilde{x}_t\}$ the state process associated with $\widetilde{\phi}$. By Assumption 5.6(c), $\hat{\phi}$ and $\widetilde{\phi}$ generate the same process except by the component $j$. Then, for $t = 1, 2, \ldots$, (5.15) gives

$$H_t(\hat{x}_t, \hat{x}_{t+1}, \xi_t) - H_t(\widetilde{x}_t, \widetilde{x}_{t+1}, \xi_t) = G_t^j(\hat{x}_t, \hat{x}_{t+1}, \xi_t) - G_t^j(\widetilde{x}_t, \widetilde{x}_{t+1}, \xi_t).$$

Actually, the latter equality also holds for $t = 0$. After summing over all $t = 0, 1, \ldots$ and taking expectations, we see that $\hat{\phi}$ is a Nash equilibrium for the game, because $j$ and $\phi^j$ are arbitrary. Therefore, (5.6) is a DPG. □

In Theorem 5.10, below, we show that Assumption 5.8, due to Dechert [9] for the deterministic case, is equivalent to (5.16)–(5.17) which is analogous to condition (5.5) given by Slade [27].

**Theorem 5.10.** *Suppose that the game* (5.6) *satisfies Assumption 5.6. Then the following conditions are equivalent:*

*(i) Assumption 5.8 holds;*

*(ii) For each $j = 1, \ldots, n$, there exist functions $H_t$ and $g_t^j$ $(t = 0, 1, \ldots)$ of class $\mathcal{C}^2$ such that*

$$G_0^j(x,y,s) = H_0(x,y,s) + g_0^j(x, y^{-j}, s) \quad \forall (x,y,s), \qquad (5.16)$$

*and for $t = 1, 2, \ldots$*

$$G_t^j(x,y,s) = H_t(x,y,s) + g_t^j(x^{-j}, y^{-j}, s) \quad \forall (x,y,s). \qquad (5.17)$$

*Proof of (i) ⇒ (ii).* Suppose that Assumption 5.8 holds. Define $H_t$, for $t = 0, 1, \ldots$, by (5.10)–(5.11). We claim that $G_t^j(x, y, s) - H_t(x, y, s)$ does not depend on $x^j$ nor on $y^j$, for each $t = 1, 2 \ldots$ and $j = 1, \ldots, n$. Indeed, let

$$F_t := (G_t^1 - H_t, \ldots, G_t^n - H_t) \quad t = 1, 2, \ldots.$$

Then $\partial_x F_t = b_t - \partial_x H_t$, but (5.11) yields $\partial_x H_t = b_t$. Hence

$$\partial_x F_t(x, y, s) = 0 \quad \forall (x, y, s), \tag{5.18}$$

for every $t = 1, 2, \ldots$. On the other hand, by (5.11), (5.13), and Assumption 5.8(a),

$$\begin{aligned} \partial_y F_t(x, y, s) &= a_t(x, y, s) - a_t(\overline{x}, y, s) - \partial_y \int_{\overline{x}}^{x} b_t(w, y, s)\, dw \\ &= a_t(x, y, s) - a_t(\overline{x}, y, s) - \int_{\overline{x}}^{x} \partial_y b_t(w, y, s)\, dw \\ &= a_t(x, y, s) - a_t(\overline{x}, y, s) - \int_{\overline{x}}^{x} [\partial_x a_t(w, y, s)]^*\, dw \\ &= 0 \end{aligned} \tag{5.19}$$

for every $t = 1, 2, \ldots$. From (5.18) and (5.19) we observe that the difference $G_t^j(x, y, s) - H_t(x, y, s)$ does not depend on $(x^j, y^j)$, and so there exists a function $g_t^j$ such that

$$G_t^j(x, y, s) - H_t(x, y, s) = g_t^j(x^{-j}, y^{-j}, s) \quad \forall (x, y, s),$$

for each $j = 1, \ldots, n$ and $t = 1, 2, \ldots$. This proves (5.17). A similar argument can be used to prove (5.16).

*Proof of (ii) ⇒ (i).* Assume that there exist functions $H_t$ and $g_t^j$ verifying (5.16)–(5.17). We next verify Assumption 5.8. Recall Remark 5.7.

(a) Since $H_t$ is of class $\mathcal{C}^2$, then $\partial^2 H_t / \partial x^i \partial y^j = \partial^2 H_t / \partial y^j \partial x^i$. Hence $\partial_x a_t = [\partial_y b_t]^*$.

(b) Note that $\partial a_t^i / \partial y^j = \partial a_t^j / \partial y^i$, for every $i, j = 1, \ldots, n$, then $a_t(x, \cdot, s)$ is exact. Analogously for $b_t(\cdot, y, \xi_t)$.

This proves the theorem. □

**Remark 5.11.** Slade [27] claims that Theorem 5.5, stated above, can be extended to dynamic games with open–loop strategies. In fact, by virtue of Theorem 5.10, it turns out that Theorem 5.9 is valid not only for open–loop strategies; it also holds for strategies as in Assumption 5.6(c).

Note that the stochastic lake game (SLG) in Example 2.2 has only one state variable; hence, this game cannot be studied as a DPG using Theorem 5.9. However, as we will see below, it belongs to another class of DPGs.

## 5.3 A class of open–loop DPGs

Consider a game with dynamics and reward functions given by (2.1) and (2.2), respectively. The state spaces $\{X_t\}$ are subsets of $\mathbb{R}^m$ and each control set $U_j \subseteq \mathbb{R}^{m_j}$ for $j = 1, \ldots, n$. Finally, we consider the sets $\Psi^j$ $(j = 1, \ldots, n)$ of open–loop multi–strategies. In reduced form, the game can be expressed as:

$$(\{X_t\}, \{\xi_t\}, \{U_j | j \in J\}, \{f_t\}, \{r_t^j | j \in J\}, \{\Psi^j | j \in J\}). \tag{5.20}$$

It should be noted that in Subsection 5.2 we restricted ourselves to games with state variables of $n$ components (see Assumption 5.6(a)), whereas the present subsection is devoted to games in a more general setting. Nonetheless, if we are looking for DPGs, the condition $(ii)$ (due to Slade [27]) in Theorem 5.10 suggests that the reward functions should be additively separable.

**Theorem 5.12.** *Suppose that the functions* $\{r_t^j\}$ *in* (5.20) *are of the form*

$$r_t^j(x_t, u_t) = H_t(x_t, u_t) + g_t^j(u_t^{-j}), \quad j = 1, \ldots, n, \ t = 0, 1, \ldots, \tag{5.21}$$

*for some functions* $g_t^j$ *and* $H_t$*. Then the game* (5.20) *is an open–loop potential game.*

*Proof.* We need to specify an OCP. Thus consider the same components of (5.20) with exception of the reward functions. Define the performance index $\mathbb{E} \sum_{t=0}^{\infty} H_t(x_t, u_t)$ with $H_t$ given by (5.21).

We are going to show that an optimal policy to the latter OCP is indeed an OLNE. Let $\hat{\psi} = (\hat{\psi}^1, \ldots, \hat{\psi}^n)$ be an (open–loop) optimal policy to the OCP and denote by $\{\hat{x}_t\}$ the corresponding state process. Fix a player $j$ and define

$$\widetilde{\psi} = (\hat{\psi}^1, \ldots, \hat{\psi}^{j-1}, \widetilde{\psi}^j, \hat{\psi}^{j+1}, \ldots, \hat{\psi}^n),$$

for some strategy $\widetilde{\psi}^j \in \Psi^j$. Denote by $\{\widetilde{x}_t\}$ the process associated to $\widetilde{\psi}$.

Then $\mathbb{E} \sum_{t=0}^{\infty} H_t(\hat{x}_t, \hat{\psi}(t)) \geq \mathbb{E} \sum_{t=0}^{\infty} H_t(\widetilde{x}_t, \widetilde{\psi}(t))$, and therefore (5.21) yields

$$\mathbb{E} \sum_{t=0}^{\infty} \left[ r_t^j(\hat{x}_t, \hat{\psi}(t)) - g_t^j(\hat{\psi}^{-j}(t)) \right] \geq \mathbb{E} \sum_{t=0}^{\infty} \left[ r_t^j(\widetilde{x}_t, \widetilde{\psi}(t)) - g_t^j(\widetilde{\psi}^{-j}(t)) \right].$$

Equivalently,

$$\mathbb{E} \sum_{t=0}^{\infty} r_t^j(\hat{x}_t, \hat{\psi}(t)) \geq \mathbb{E} \sum_{t=0}^{\infty} r_t^j(\widetilde{x}_t, \widetilde{\psi}(t)).$$

Therefore, $\hat{\psi}$ is an OLNE for the game (5.20), that is, (5.20) is an open–loop potential game. □

**Remark 5.13.** From the proof of Theorem 5.12, it can be observed that the function $g_t^j$ cannot depend on the state variable $x_t$. This is because the state $x_t$ is determined by the actions $u_{t-1}$ of all players, including player $j$ (see (2.1)).

**Corollary 5.14.** *Suppose that the functions* $\{r_t^j\}$ *in* (5.20) *are of the form*

$$r_t^j(x_t, u_t) = L_t(x_t, u_t) + L_t^j(u_t^j), \quad j = 1, \dots, n, \; t = 0, 1, \dots, \tag{5.22}$$

*for some functions* $L_t^j$ *and* $L_t$. *Then the game* (5.20) *is an open–loop potential game.*

*Proof.* The result follows from Theorem 5.12 and the equality

$$L_t(x_t, u_t) + L_t^j(u_t^j) = \left[L_t(x_t, u_t) + \sum_{i=1}^{n} L_t^i(u_t^i)\right] - \sum_{i \neq j} L_t^i(u_t^i). \tag{5.23}$$

□

**Example 5.15.** Clearly, by Corollary 5.14, the SLG given in Example 2.2 is an open–loop potential game. From (5.23) and Theorem 5.12, a performance index for the associated OCP is given by (2.7)–(2.8). More precisely, let $H(u, x)$ be as in (2.8), with $u = (u^1, \dots, u^n)$, and let

$$r_j(x, u) := v_j(u^j) - x^2$$

be the function within brackets in (2.6). Then (comparing with Definition 5.3) we have

$$\frac{\partial H}{\partial u_j} = \frac{\partial r_j}{\partial u_j} \quad \forall\, j = 1, \dots, n.$$

# 6 A class of games for which Nash equilibria are Pareto solutions

In this subsection we consider Pareto solutions to dynamic games. That is, a (Markov or open–loop) multi–strategy $\phi$ is called a *Pareto solution* for the game (5.20) if $\phi$ maximizes the convex combination

$$\mathbb{E} \sum_{t=0}^{\infty} [\lambda_1 r_t^1(x_t, u_t) + \ldots + \lambda_n r_t^n(x_t, u_t)], \tag{6.1}$$

subject to (2.1), for some $\lambda_j > 0$ ($j = 1, \dots, n$) such that $\lambda_1 + \ldots + \lambda_n = 1$.

**Theorem 6.1.** *Suppose that the functions* $\{r_t^j\}$ *in* (5.20) *are of the form*

$$r_t^j(x_t, u_t) = g_t^j(u_t^j), \quad j = 1, \ldots, n, \; t = 0, 1, \ldots. \tag{6.2}$$

*Then the game* (5.20) *is an open–loop potential game. Moreover, each open–loop Pareto solution to the game* (5.20) *is also an OLNE.*

*Proof.* By Corollary 5.14, it is clear that a game with reward functions as in (6.2) is an open–loop potential game. It remains to show that an open–loop Pareto solution is also an OLNE.

Let $\hat{\psi} = (\hat{\psi}^1, \ldots, \hat{\psi}^n)$ be an open–loop Pareto solution to the game (5.20). As in the proof of Theorem 5.12, denote by $\{\hat{x}_t\}$ and $\{\widetilde{x}_t\}$ the corresponding processes associated to $\hat{\psi}$ and $\widetilde{\psi}$, respectively. Recall that

$$\widetilde{\psi} = (\hat{\psi}^1, \ldots, \hat{\psi}^{j-1}, \widetilde{\psi}^j, \hat{\psi}^{j+1}, \ldots, \hat{\psi}^n),$$

for some strategy $\widetilde{\psi}^j \in \Psi^j$. Then

$$\mathbb{E} \sum_{t=0}^{\infty} \left[ \lambda_1 g_t^1(\hat{\psi}^1(t)) + \ldots + \lambda_n g_t^n(\hat{\psi}^n(t)) \right] \geq \mathbb{E} \sum_{t=0}^{\infty} \left[ \lambda_1 g_t^1(\widetilde{\psi}^1(t)) + \ldots + \lambda_n g_t^n(\widetilde{\psi}^n(t)) \right],$$

for some $\lambda_j > 0$ $(j = 1, \ldots, n)$ such that $\lambda_1 + \ldots + \lambda_n = 1$. Since $\hat{\psi}^i = \widetilde{\psi}^i$ for every $i \neq j$ and each $\lambda_i > 0$, the latter inequality becomes

$$\mathbb{E} \sum_{t=0}^{\infty} g_t^j(\hat{\psi}^j(t)) \geq \mathbb{E} \sum_{t=0}^{\infty} g_t^j(\widetilde{\psi}^j(t)).$$

Because $j$ and $\widetilde{\psi}^j$ are arbitrary, $\hat{\psi}$ is an OLNE for the game (5.20). This yields the desired conclusion. □

**Remark 6.2.** It is important to note that differentiability and convexity hypotheses are not required in Theorems 5.12 and 6.1 nor in Corollary 5.14. In addition, these results are also valid for the finite horizon case as well as the deterministic problems. Moreover, with the appropriate changes, they can be stated (and proved in the same way as above) in continuous–time.

**Example 6.3.** Let us go back to the game in Example 4.6. By Theorem 6.1, every Pareto solution to the fish war game is also an OLNE. The Pareto solutions to this game are explicitly found in González–Sánchez and Hernández–Lerma [13]. It can be verified that, in fact, each Pareto solution solves the difference equation (4.10) obtained in Example 4.6. Moreover, by (6.1), there exist infinitely many Pareto solutions which, by Theorem 6.1, are also OLNE.

## 7 Concluding remarks

Sufficient conditions to identify MNE and OLNE, by following the EE approach, were given in Theorems 4.2 and 4.5, respectively. As we mentioned in the Introduction, one of our main objectives was to identify DPGs by generalizing the procedure of Dechert and O'Donnell for the SLG. By using stochastic inverse problems, we identified a first class of DPGs in Theorem 5.9. However, the SLG is not included in this class of games. Nonetheless, the SLG belongs to a second class of DPGs given in Theorem 5.12.

In Theorem 5.10 we proved the equivalence between the conditions found, independently, by Dechert [9] and Slade [27]. The approach followed by Dechert is similar to the Monderer and Shapley's (see Theorem 5.4) but different from Slades's approach. Using Slade's characterization (Theorem 5.10(ii)), some dynamic potential games with simple reward functions can be easily identified. However when the reward functions are more complicated, Dechert's procedure is more efficient and it has an explicit formula for the potential function. Slade [27] points out that Theorem 5.5 can be directly extended to dynamic games with open–loop strategies. This is true for games like (5.6)—see (5.17)—; however in dynamic games with explicit control variables as in (5.20), it is necessary to be more specific about the separability on the reward functions. See Remarks 5.11 and 5.13.

A subclass of DPGs was characterized in Theorem 6.1. A *noncooperative* OLNE of a game in this subclass is also a (cooperative) Pareto solution. The fish war game, described in Example 4.6, belongs to this subclass of DPGs; see Example 6.3. We see that the fish war game has infinitely many OLNE. Hence uniqueness of OLNE does not hold in general despite having strictly concave reward functions.

Aumann [1, 2] introduced an important equilibrium concept known as *correlated equilibrium* which generalizes the Nash equilibrium concept. There are some results, in the static case, relating correlated equilibria and potential games; see [23] or [30]. It would be an interesting problem to extend those results to the dynamic case.

We have not considered continuous–time problems, but we believe that some of our results in Section 3 can be extended to differential games. It is also important to note that continuous–time problems can be aproximated by discrete–time problems.

An immediate advantage of identifying dynamic potential games is that some results on existence of Nash equilibria can be given, since it is easier to establish existence in OCPs than in dynamic games. Another advantage is that OCPs can be solved numerically, so it is possible obtain Nash equilibria for potential games. Actually, this the case for the SLG where Dechert and O'Donnell find Nash equilibria using numerical methods applied to the Bellman equation. In the discounted stationary case, the EE can be solved numerically; see, for instance, Maldonado and Moreira [18, 19] and Judd [15] and the references therein. Extending those methods to nonstationary control problems and games seems to be an important open problem. In particular, numerical methods for discrete–time problems can be used to approximate solutions to continuous–time problems.

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